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Accurate Approximation in Weighted Maximum Norm by Interpolation

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1. THE MAIN IDEA

In this paper we will compare the weighted maximum error achieved by approximation defined by interpolation with weighted minimax criteria.

We use the following notation:

$p^*(x)$ is the interpolation polynomial on the net $\{x_i\}_{i=1}^n$;

$\hat{p}(x)$ (of degree $n-1$) is the weighted minimax approximation defined by

$$\max_{a\leqslant x\leqslant b} \rho(x)\,|f(x)-\hat{p}(x)| \leqslant \max_{a\leqslant x\leqslant b} \rho(x)\,|f(x)-p(x)| \tag{1}$$

or

$$\|f-\hat{p}\|_\rho \leqslant \|f-p\|_\rho,$$

where $p(x)$ is any polynomial of degree less than n and $\rho(x)$ is the weight function. $\rho(x)$ is positive but can be zero at the endpoints.

The error functions are

$$e^*(x) = f(x) - p^*(x)$$

and

$$\hat{e}(x) = f(x) - \hat{p}(x),$$

with

$$M^* = \|e^*\|_\rho \qquad \text{and} \qquad \hat{M} = \|\hat{e}\|_\rho .$$

Inequality (1) immediately gives $M^* \geqslant \hat{M}$.

We first state a theorem given by Gustafson [2] which generalizes a result by Powell [6].

THEOREM 1.

$$\frac{M^*}{\hat{M}} \leqslant 1 + \max_{a\leqslant x\leqslant b} \rho(x) \sum_{i=1}^{n} \frac{|l_i(x)|}{\rho(x_i)},$$

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where $l_i(x)$ is the polynomial of degree less than n which satisfies the relations

$$l_i(x_j) = 0, \quad i \neq j \qquad \text{for} \quad j = 1, 2, \ldots, n$$
$$ = 1, \quad i = j$$

and $a < x_i < b$ for $i = 1, 2, \ldots, n$.

Proof. $\hat{e}(x) - e^*(x) = p^*(x) - \hat{p}(x)$ is a polynomial of degree less than n satisfying $\hat{e}(x_i) - e^*(x_i) = \hat{e}(x_i)$ for $i = 1, 2, \ldots, n$.

Then we have

$$\hat{e}(x) - e^*(x) = \sum_{i=1}^{n} \hat{e}(x_i)\, l_i(x)$$

and we get

$$| e^*(x)| = \left| \hat{e}(x) - \sum_{i=1}^{n} \hat{e}(x_i)\, l_i(x) \right|$$
$$= \left| \hat{e}(x) - \sum_{i=1}^{n} \frac{\rho(x_i)}{\rho(x_i)} \cdot \hat{e}(x_i)\, l_i(x) \right| \leqslant | \hat{e}(x)| + \hat{M} \sum_{i=1}^{n} \frac{| l_i(x)|}{\rho(x_i)} .$$

Multiplying by $\rho(x)$ and maximizing we get

$$M^* \leqslant \hat{M} + \hat{M} \max_{a \leqslant x \leqslant b} \rho(x) \sum_{i=1}^{n} \frac{| l_i(x)|}{\rho(x_i)} ,$$
$$\frac{M^*}{\hat{M}} \leqslant 1 + \max_{a \leqslant x \leqslant b} \rho(x) \sum_{i=1}^{n} \frac{| l_i(x)|}{\rho(x_i)} . \qquad \text{Q.E.D.}$$

Now, let the interpolation operator be L. Then the norm of L is

$$\| L \|_\rho = \max_{a \leqslant x \leqslant b} \rho(x) \sum_{i=1}^{n} \frac{| l_i(x)|}{\rho(x_i)} ,$$

the same number which occurs in Theorem 1. $\| L \|_\rho$ can immediately be applied to bound the influence of rounding errors in the function values:

$$\| p^*(x) - \tilde{p}(x)\|_\rho \leqslant \epsilon \cdot \max_{a \leqslant x \leqslant b} \rho(x) \sum_{i=1}^{n} \frac{| l_i(x)|}{\rho(x_i)} ,$$

where ϵ is an upper bound of the modulus of the rounding errors in the function values and $\tilde{p}(x)$ is the interpolation polynomial in the approximate function values. For $\rho(x) \equiv 1$ this was shown in [5].

2. A Specific Class of Weight Functions

In this section we discuss how to choose the interpolation points when the weighted maximum norm of $h(x)$ is defined as

$$\| h(x)\|_p = \max_{-1 \leqslant x \leqslant 1} (1-x)^\alpha (1+x)^\beta \, | h(x)|.$$

Our problem is to choose the net $\{x_i\}_{i=1}^n$ so that $M^*/\hat{M}$ is "close" to 1. The numerical experiments lead us to choose the zeros of $P_n^{(2\alpha-\frac{1}{2},2\beta-\frac{1}{2})}(x)$, where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree n with weightfunction $(1-x)^\alpha(1+x)^\beta$. Doing so we get a near optimal solution. In Theorem 2 an asymptotic limit value when $n \to \infty$ is given for $M^*/\hat{M}$. This limit value is independent of α and β. Note that for $\alpha = \beta = 0$ we get the well-known Chebyshev abscissas and the result of Powell [6].

To show Theorem 2 we need some lemmas. Most of these results can be found in [7]. We use the following notations:

$P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree n with weight function $(1-x)^\alpha(1+x)^\beta$,

$J_\alpha(x)$ is the Bessel function of first kind of order α,

$\Gamma(x)$ is the Gamma function,

$A(n) \sim B(n)$ in the sense that the ratio of $A(n)/B(n) \to 1$ as $n \to \infty$.

Lemma 1 [see 7, Theorem 8.21.12 and formula (4.1.1)]. *Let $\alpha > -1$ and β be arbitrary and real. Then we have*

$$\left(\sin\frac{\theta}{2}\right)^\alpha \left(\cos\frac{\theta}{2}\right)^\beta P_n^{(\alpha,\beta)}(\cos\theta) = N^{-\alpha}\frac{\Gamma(n+\alpha+1)}{n!}\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_\alpha(N\theta) + \theta^{1/2}\mathcal{O}(n^{-3/2}) \qquad \text{if } \; 0 < \theta \leqslant \pi - \epsilon$$

and

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n},$$

where $N = n + (\alpha+\beta+1)/2$; ϵ is a fixed positive number.

Lemma 2 (see [7, Theorem 7.32.4]). *Let α and β be arbitrary and real; c is a fixed positive constant. Then*

$$\left\{\frac{d}{dx} P^{(\alpha,\beta)}(x)\right\}_{x=\cos\theta} \sim \begin{cases} \theta^{-\alpha-\frac{3}{2}}\mathcal{O}(n^{1/2}) & \text{if } \; \dfrac{c}{n} \leqslant \theta \leqslant \dfrac{\pi}{2}, \\ \mathcal{O}(n^{\alpha+2}) & \text{if } \; 0 \leqslant \theta \leqslant \dfrac{c}{n}. \end{cases}$$

LEMMA 3 (see [7, Theorem 8.9.1]). *Let* $\alpha > -1$ *and* $\beta > -1$ *and let* $0 < \theta_1 < \theta_2 < \cdots < \theta_r \leqslant \pi/2$ *be zeros of* $P_n^{(\alpha,\beta)}(\cos\theta)$. *Then*

$$\left|\frac{d}{dx} P_n^{(\alpha,\beta)}(\cos\theta_\nu)\right| \sim \nu^{-\alpha-\frac{3}{2}}\mathcal{O}(n^{\alpha+2}), \qquad \nu = 1, 2,\ldots, r.$$

LEMMA 4 (see [1, formula (6.1.46)]).

$$\lim_{n\to\infty} n^{b-a}\frac{\Gamma(n+a)}{\Gamma(n+b)} = 1.$$

Note. Putting $a = 2\alpha + \frac{1}{2}$ and $b = 1$ in Lemma 4 we get

$$\lim_{n\to\infty}\frac{\Gamma(n+2\alpha+\frac{1}{2})}{n!\,n^{2\alpha-\frac{1}{2}}} = 1 \qquad \text{since} \qquad \Gamma(n+1) = n!$$

LEMMA 5 (see [7, formula (1.71.7)]). *The following asymptotic formula holds*:

$$J_\alpha(z) = \left(\frac{2}{\pi z}\right)^{1/2}\cos\left(z - \alpha\frac{\pi}{2} - \frac{\pi}{4}\right) + \mathcal{O}(z^{-3/2}) \qquad as \quad z \to \infty.$$

Note that

$$|J_\alpha(z)| \leqslant \left(\frac{2}{\pi|z|}\right)^{1/2} + |\mathcal{O}(z^{-3/2})| \sim \left(\frac{2}{\pi}\right)^{1/2}|z|^{-1/2}.$$

LEMMA 6 (see [7, Theorem 8.9.1, formula (8.9.5), Theorem 8.1.2, and formula (8.1.4)]). *Let* $\alpha > -1$, $\beta > -1$, *and let* $0 < \theta_1 < \theta_2 < \cdots < \theta_n < \pi$ *be the zeros of* $P_n^{(\alpha,\beta)}(\cos\theta)$. *Then* $\theta_i = (i\pi + \mathcal{O}(1))/n$, *with* $\mathcal{O}(1)$ *being uniformly bounded for all values of* $i = 1, 2,\ldots, n$; $n = 1, 2, 3,\ldots$. *If* $\alpha = 0$, *then*

$$\theta_i = \frac{(i-\frac{1}{4})\pi + o(1)}{n}, \qquad i = 1, 2,\ldots, n.$$

Furthermore,

$$\frac{d}{d\theta}\{P_n^{(\alpha,\beta)}(\cos\theta)\} \sim \mathcal{O}(n^{1/2})\left(\sin\frac{\theta}{2}\right)^{-\alpha-\frac{1}{2}}\left(\cos\frac{\theta}{2}\right)^{-\beta-\frac{1}{2}}.$$

The zeros from a fixed interval in the interior of $[0, \pi]$ can be written in a more precise form.

LEMMA 7 (see [7, formulas (8.9.8) and (8.8.1)]). *Let* $\{\cos\theta_i\}_{i=1}^n$ *be the zeros of* $P_n^{(\alpha,\beta)}(x)$ *in the interval* $[\epsilon, \pi-\epsilon]$, *where* ϵ *is a fixed positive number and* $\alpha > -1$, $\beta > -1$. *Then*

$$\theta_i = ((i-\tfrac{1}{2})\pi - \gamma + k\pi + \epsilon_n)/N,$$

where

$$\gamma = -(\alpha + \tfrac{1}{2})(\pi/2),$$
$$N = n + (\alpha + \beta + 1)/2,$$
k *is an integer independent of* i *and* n,
$$\epsilon_n \to 0 \quad \text{when} \quad n \to \infty.$$

Furthermore,

$$\left| \frac{d}{d\theta} \{P_n^{(\alpha,\beta)}(\cos\theta)\} \right|_{\theta=\theta_i} \sim \pi^{-1/2} n^{1/2} \left(\sin\frac{\theta_i}{2}\right)^{-\alpha-\frac{1}{2}} \left(\cos\frac{\theta_i}{2}\right)^{-\beta-\frac{1}{2}}.$$

Note that

$$|\sin(N\theta_i + \gamma)| = \left|\sin\left(N \cdot \frac{(i-\frac{1}{2})\pi - \gamma + k\pi + \epsilon_n}{N} + \gamma\right)\right| \sim 1.$$

LEMMA 8. *Let* $\{\cos\theta_i\}_{i=1}^n$ *be the zeros of* $P_n^{(\alpha,\beta)}(x)$ *and let* $\theta \in (0, \pi)$ *be fixed. Then*

$$\sum_{|\cos\theta - \cos\theta_i| \geqslant 1/n} \left| \frac{\sin\theta_i}{\cos\theta - \cos\theta_i} \right| \sim \frac{2}{\pi} n \ln n.$$

Proof. For brevity we write c for $\cos\theta$, c_i for $\cos\theta_i$, and s_i for $\sin\theta_i$. Let δ be a positive fixed number such that $-1 + \delta < \cos\theta < 1 - \delta$. Then for $n > 1/\delta$

$$\sum_{|c-c_i| \geqslant 1/n} \left| \frac{s_i}{c - c_i} \right| = \sum_{|c-c_i| > \delta} \left| \frac{s_i}{c - c_i} \right| + \sum_{1/n \leqslant |c-c_i| \leqslant \delta} \left| \frac{s_i}{c - c_i} \right|$$
$$= \mathcal{O}(n) + \sum_{1/n \leqslant |c-c_i| \leqslant \delta} \left| \frac{s_i}{c - c_i} \right|.$$

Now we know from Lemma 7 that $\theta_i = (i\pi + k_1 + \epsilon_n(i))/(n + k_2)$, where k_1 and k_2 are independent of i and n and $\epsilon_n(i) \to 0$ when $n \to \infty$.

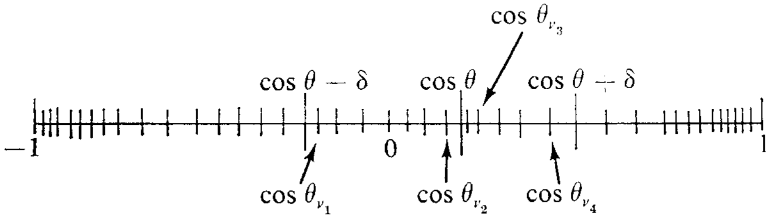
We now determine

$$S_\theta = \sum_{1/n \leqslant |c-c_i| \leqslant \delta} \left| \frac{s_i}{c - c_i} \right|$$

using the Euler–MacLaurin formula with a strict error bound

$$\left| \sum_{i=1}^{n} f(i) - \int_1^n f(t)\, dt \right| \leqslant \frac{1}{2} |f(1) + f(n)| + \frac{2}{12} \int_1^n \left| \frac{d^2 f(t)}{dt^2} \right| dt.$$

We split the interval $1/n \leqslant |c - c_i| \leqslant \delta$ into two parts such that



$$S_\theta = \sum_{i=\nu_1}^{\nu_2} s_i/(c - c_i) + \sum_{i=\nu_3}^{\nu_4} s_i/(c_i - c).$$

Now,

$$\frac{s_i}{c - c_i} - \frac{\sin((i\pi + k_1)/n)}{c - \cos((i\pi + k_1)/n)} = o(1) \cdot \frac{\sin((i\pi + k_1)/n)}{c - \cos((i\pi + k_1)/n)},$$

which can be seen by writing

$$s_i = \sin\frac{i\pi + k_1}{n + k_2} \cdot \cos\frac{\epsilon_n(i)}{n + k_2} + \cos\frac{i\pi + k_1}{n + k_2} \cdot \sin\frac{\epsilon_n(i)}{n + k_2}$$

using the McLaurin expansions of

$$\cos\frac{\epsilon_n(i)}{n + k_2} \qquad \text{and} \qquad \sin\frac{\epsilon_n(i)}{n + k_2}$$

and finally using the fact that $c - c_i > 1/n$.

Hence

$$\begin{aligned}\sum_{i=\nu_1}^{\nu_2} \frac{s_i}{c - c_i} &= (1 + o(1)) \sum_{i=\nu_1}^{\nu_2} \frac{\sin((i\pi + k_1)/n)}{\cos\theta - \cos((i\pi + k_1)/n)} \\ &= (1 + o(1)) \left(\int_{\nu_2}^{\nu_1} \frac{\sin((t\pi + k_1)/n)}{\cos\theta - \cos((t\pi + k_1)/n)}\, dt + R(n)\right) \\ &\sim (n/\pi)(\ln(\cos\theta - \cos\theta_{\nu_1}) - \ln(\cos\theta - \cos\theta_{\nu_2}))\end{aligned}$$

while

$$\begin{aligned}|R(n)| \leqslant \frac{1}{2}&\left(\frac{\sin((\nu_1\pi + k_1)/n)}{\cos\theta - \cos((\nu_1\pi + k_1)/n)} + \frac{\sin((\nu_2\pi + k_1)/n)}{\cos\theta - \cos((\nu_2\pi + k_1)/n)}\right) \\ &+ \frac{2}{12}\int_{\nu_1}^{\nu_2} \left|\frac{d^2}{dt^2}\left\{\frac{\sin((t\pi + k_1)/n)}{\cos\theta - \cos((t\pi + k_1)/n)}\right\}\right| dt \sim \mathcal{O}(n)\end{aligned}$$

since $|c - c_i| \geqslant 1/n$, and hence negligible.

In the same way we get

$$\sum_{i=\nu_3}^{\nu_4} \frac{s_i}{c_i - c} \sim \frac{n}{\pi} (\ln(\cos\theta - \cos\theta_{\nu_4}) - \ln(\cos\theta - \cos\theta_{\nu_3}))$$

and finally

$$S_\theta \sim \frac{n}{\pi} \ln \left(\frac{\cos\theta - \cos\theta_{\nu_1}}{\cos\theta - \cos\theta_{\nu_2}} \cdot \frac{\cos\theta - \cos\theta_{\nu_4}}{\cos\theta - \cos\theta_{\nu_3}}\right)$$

$$\sim \frac{n}{\pi} \ln \frac{\delta \cdot \delta}{(1/n)\cdot(1/n)} \sim \frac{2}{\pi} n \ln n. \qquad \text{Q.E.D.}$$

THEOREM 2. *Let the netpoints* $\{x_i\}_{i=1}^n$ *be the zeros of* $P_n^{(2\alpha-\frac{1}{2},2\beta-\frac{1}{2})}(x)$ *and* $\rho(x) = (1-x)^\alpha(1+x)^\beta$, $\alpha \geqslant 0$, *and* $\beta \geqslant 0$. *Then*

$$\max_{-1\leqslant x\leqslant 1} \rho(x) \sum_{i=1}^n \frac{|l_i(x)|}{\rho(x_i)} \sim \frac{2}{\pi} \ln n.$$

Proof. Since $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$ we only study x belonging to $[0, 1]$. Put $x = \cos\theta$ which gives $\rho(x) = 2^{\alpha+\beta}\sin^{2\alpha}(\theta/2)\cos^{2\beta}(\theta/2)$, where $\theta \in [0, \pi/2]$. We put

$$K(x) = \rho(x) \sum_{i=1}^n \frac{|l_i(x)|}{\rho(x_i)}$$

$$= \rho(x)\,|P_n^{(2\alpha-\frac{1}{2},2\beta-\frac{1}{2})}(x)| \sum_{i=1}^n \frac{1}{|x - x_i|\,\rho(x_i)\,|P_n'^{(2\alpha-\frac{1}{2},2\beta-\frac{1}{2})}(x_i)|}.$$

We first look at $A(x) = \rho(x)\,|P_n^{(2\alpha-\frac{1}{2},2\beta-\frac{1}{2})}(x)|$ when $0 \leqslant x < 1$.

$$A(x) = 2^{\alpha+\beta}\sin^{2\alpha}\frac{\theta}{2}\cos^{2\beta}\frac{\theta}{2}\,|P_n^{(2\alpha-\frac{1}{2},2\beta-\frac{1}{2})}(x)|$$

$$= 2^{\alpha+\beta}\sin^{1/2}\frac{\theta}{2}\cos^{1/2}\frac{\theta}{2}\left|\frac{\Gamma(n+2\alpha+\frac{1}{2})}{N^{2\alpha-\frac{1}{2}}n!}\right.$$

$$\left.\cdot\left(\frac{\theta}{\sin\theta}\right)^{1/2} J_{2\alpha-\frac{1}{2}}(N\theta) + \mathcal{O}(n^{-3/2})\right|$$

$$= 2^{\alpha+\beta-\frac{1}{2}}\sin^{1/2}\theta\left|\frac{(n+2\alpha-\frac{1}{2})!}{(n+\alpha+\beta)^{2\alpha-\frac{1}{2}}\,n!}\cdot\frac{\theta^{1/2}}{\sin^{1/2}\theta}\right.$$

$$\left.\cdot J_{2\alpha-\frac{1}{2}}(N\theta) + \mathcal{O}(n^{-3/2})\right| \tag{2}$$

according to Lemma 1, where $N = n + \alpha + \beta$. If we now use Lemmas 4 and 5 we get

$$\max_{0 \leqslant x < 1} A(x) \sim 2^{\alpha+\beta-\frac{1}{2}}\theta^{1/2}(2/\pi)^{1/2}(1/N\theta)^{1/2} = 2^{\alpha+\beta}\pi^{-1/2}n^{-1/2}.$$

Putting $K(x) = A(x) \cdot B(x)$, we next look at $B(x) = \sum_{i=1}^{n} 1/B_i(x)$, where

$$\begin{aligned} B_i(x) &= | x - x_i | \rho(x_i) | P_n'^{(2\alpha-\frac{1}{2},2\beta-\frac{1}{2})}(x_i)| \\ &= | x - x_i | 2^{\alpha+\beta} \sin^{2\alpha}(\theta_i/2) \cos^{2\beta}(\theta_i/2) \sin^{-1} \theta_i \\ &\quad \cdot \left| \frac{d}{d\theta} P_n^{(2\alpha-\frac{1}{2},2\beta-\frac{1}{2})}(\cos \theta) \right|_{\theta=\theta_i} . \end{aligned}$$

Now let δ be a positive fixed number and let $x \in [0, 1 - \delta]$.

We first study $B_i(x)$ when $| x - x_i | > \delta$. Then $B_i(x) \gtrsim \delta \cdot \sin^{-1} \theta_i \mathcal{O}(n^{1/2})$ according to Lemma 6. Hence $\sum_{|x-x_i|>\delta} (1/B_i(x)) \lesssim \mathcal{O}(n^{1/2})$. Next we study $B_i(x)$ for $1/n \leqslant | x - x_i | \leqslant \delta$. Using Lemma 7 we get

$$B_i(x) \sim | x - x_i | 2^{\alpha+\beta} \sin^{-1} \theta_i n^{1/2}\pi^{-1/2}.$$

Now Lemma 8 gives

$$\begin{aligned} \sum_{1/n \leqslant |x-x_i| \leqslant \delta} \frac{1}{B_i(x)} &\sim 2^{-(\alpha+\beta)}n^{-1/2}\pi^{+1/2} \cdot \frac{2}{\pi} n \ln n \\ &\sim 2^{-(\alpha+\beta)+1}\pi^{-1/2}n^{1/2} \ln n. \end{aligned}$$

It remains to study $\sum_{|x-x_i|<1/n} (1/B_i(x))$. Since $A(x)$ is small for these values we consider $A(x) \sum_{|x-x_i|<1/n} (1/B_i(x))$. Trivially this term is zero for $x = x_i$, $i = 1, 2,..., n$. Now Lemma 7 gives

$$\theta_i = \frac{i\pi + k_1 + \epsilon_n(i)}{n + k_2},$$

k_1 and k_2 are independent of i, and n and $\epsilon_n(i) \to 0$ when $n \to \infty$, and hence there is an n_0 such that $| x - x_i | < 1/n$ can hold for at most $\sin^{-1} \delta$ terms if $n > n_0$.

We study any of these, say $A(x)/B_p(x)$, and use the Taylor expansion of $P(x) = P_n^{(2\alpha-\frac{1}{2},2\beta-\frac{1}{2})}(x)$ and $\rho(x)$.

$$\begin{aligned} \frac{A(x)}{B_p(x)} &= \frac{\rho(x) | P(x)|}{\rho(x_p) | x - x_p | | P'(x_p)|} \\ &= \frac{\rho(x_p) + (x - x_p) \rho'(\xi)}{\rho(x_p)} \cdot \frac{| P(x_p) + (x - x_p) P'(\eta)|}{| x - x_p | | P'(x_p)|} \\ &\leqslant (1 + \mathcal{O}(1/n)) \cdot \mathcal{O}(1) \sim \mathcal{O}(1), \end{aligned}$$

where $|x_p - \xi| < 1/n$ and $|x_p - \eta| < 1/n$, since $P(x_p) = 0$ and according to Lemmas 2, 3, and 7,

$$P'(\cos\theta) \sim \theta^{-2\alpha-1} \cdot \mathcal{O}(n^{1/2}), \qquad \text{for} \quad 1/n \leqslant \theta \leqslant \pi/2,$$
$$P'(\cos\theta_p) \sim p^{-2\alpha-1}\mathcal{O}(n^{2\alpha+3/2}),$$

and hence

$$\frac{P'(\eta)}{P'(x_p)} \sim \frac{\mathcal{O}(n^{1/2})}{((p\pi + k_1 + \epsilon_n)/n)^{2\alpha+1}} \cdot \frac{p^{2\alpha+1}}{n^{2\alpha+\frac{3}{2}}} \sim \mathcal{O}(1).$$

Thus

$$\sum_{|x-x_i|\leqslant 1/n} \frac{A(x)}{B_i(x)} \sim \mathcal{O}(1)$$

and

$$\max_{0\leqslant x\leqslant 1-\delta} K(x) \sim 2^{\alpha+\beta}\pi^{-1/2}n^{-1/2}(\mathcal{O}(n^{1/2}) + 2^{-(\alpha+\beta)+1}\pi^{-1/2}n^{1/2}\ln n) + \mathcal{O}(1)$$
$$\sim \frac{2}{\pi}\ln n.$$

Now since δ is an arbitrary positive number we have shown the asymptotic behavior for $x \in [0, 1)$. It only remains to study $K(x)$ for $x = 1$. If $\alpha > 0$ then $K(1) = 0$ since $\rho(1) = (1 - 1)^\alpha(1 + 1)^\beta$. If $\alpha = 0$ then we know by Lemmas 1 and 4 that

$$A(1) = 2^\beta \binom{n - \frac{1}{2}}{n} = 2^\beta \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)\,\Gamma(\frac{1}{2})} \sim 2^\beta\pi^{-1/2}n^{-1/2}$$

and by Lemma 6 that $\theta_i = (i\pi + k + o(1))/n$ for *all* θ_i and hence Lemma 8 holds for $x \in [0, 1]$ and $K(1) \lesssim (2/\pi)\ln n$. This ends the proof.

Now we can see why it is fruitful to choose the zeros of $P^{(2\alpha-\frac{1}{2},2\beta-\frac{1}{2})}(x)$ instead of those of $P^{(2\alpha,2\beta)}(x)$ which were proposed in [4].

The proof of Theorem 2, formula (2), shows that the factor $2^{1/2}\sin^{1/2}(\theta/2)\cos^{1/2}(\theta/2) = \sin^{1/2}\theta$ cancels the same factor in the denominator of the asymptotic behavior of $P^{(2\alpha-\frac{1}{2},2\beta-\frac{1}{2})}(x)$ which would otherwise contribute a factor $\mathcal{O}(n^{1/2})$ to the result. In Section 3 we can compare the estimates using the two different sets of abscissas.

3. Numerical Application

We consider the same example as in [3, 4], numerical computation of the Fourier transform,

$$G(w) = \operatorname{Re}\left\{e^{-0.5iw}\int_{0.5}^{\infty} e^{iwt}f(t)\,dt\right\},$$
$$f(t) = \frac{1}{1 + \ln(t + 1)}.$$

f is completely monotonic in the interval $(e^{-1} - 1, \infty)$. According to a well-known theorem by Bernstein we can then write

$$f(t) = \int_0^\infty e^{-x(t+0.5)}\, d\alpha(x), \qquad t \geqslant -0.5 > e^{-1} - 1, \qquad \alpha \text{ increasing};$$

see [8, Theorem 12a].

Hence

$$\int_0^\infty |\, d\alpha(x)| = \int_0^\infty d\alpha(x) = f(-0.5).$$

We assume that the numerical values of f are given at equidistant points $t_1, t_2, \ldots, t_n$, where $t_j = 0.5 + jh$. As shown in [3] we can write

$$G(w) = \int_0^1 \psi(\lambda; w)\, \rho(\lambda)\, d\beta(\lambda),$$

$$\int_0^1 \lambda^{j-1} \rho(\lambda)\, d\beta(\lambda) = f(t_j), \qquad j = 1, 2, \ldots, n,$$

where

$\psi(\lambda; w) = \operatorname{Re}\{-h/(\ln \lambda + ihw)\}$,

$\rho(\lambda) = \lambda^{1/h}$,

β is of bounded variation over [0, 1] but is not assumed to be known as an analytical expression,

$$\int_0^1 |\, d\beta(\lambda)| = \int_0^1 d\beta(\lambda) = f(-0.5).$$

We select n points $\{\lambda_k\}_{k=1}^n$ and construct a mechanical quadrature rule

$$G(w) \approx \sum_{k=1}^n m_k \psi(\lambda_k; w),$$

where the weights m_k are determined such that

$$\sum_{k=1}^n m_k \lambda_k^{j-1} = f(t_j), \qquad j = 1, 2, \ldots, n.$$

Our strategy for choosing $\{\lambda_k\}_{k=1}^n$ is independent of ψ and determined by ρ only. Let Q be the polynomial of degree less than n which interpolates ψ at $\{\lambda_k\}_{k=1}^n$. We want to make

$$\max_{0 \leqslant \lambda \leqslant 1} \rho(\lambda)\, |\, \psi(\lambda; w) - Q(\lambda)|, \qquad \text{small for a fixed } w.$$

Since $\rho(\lambda) = \lambda^{1/h}$, $\lambda \in [0, 1]$ we use the zeros of the nth degree transformed Jacobi polynomial corresponding to the weightfunction $\lambda^{2/h-\frac{1}{2}}$ in accordance with Section 2.

Numerical experiments indicate that the gain in using the actual minimizing polynomial is quite modest, as illustrated in the Table I.

TABLE I

n	Improvement factor		
	1	2	3
2	2.7	3.7	5.1
4	3.1	5.1	3.2
6	3.4	6.2	3.2
8	3.5	7.1	3.1
10	3.7	7.9	10.6
12	3.8	8.6	11.8

Table I shows estimates of the improvement factor $M^*/\hat{M}$, defined in Theorem 1. In columns 1 and 2 the abscissas are the zeros of the nth-degree transformed Jacobi polynomial with weightfunction $\lambda^{2/h-\frac{1}{2}}$ and $\lambda^{2/h}$, respectively, and the improvement factor is calculated from Theorem 1. Column 3 shows the results of the theory in [4]. Here the improvement factor is computed as

$$\| \psi - Q \|_\rho / \min_{1 \leqslant j \leqslant n+1} | e(\xi_j)|,$$

where

$$e(\lambda) = \rho(\lambda)(\psi(\lambda; w) - Q(\lambda))$$

and $\{\xi_j\}_{j=1}^{n+1}$ are points such that

(i) $| e(\xi_j)|$ is a local maximum of the error curve,

(ii) $e(\xi_j)\, e(\xi_{j+1}) < 0, j = 1, 2,..., n.$

Observe that the estimate according to Theorem 1 is independent of ψ in contrast to the estimate suggested in [4].

We report our results for the case $w = 10$, $h = 1/8$ in Table I. As expected, the improvement factors in column 2 are much larger than those in column 1, since the asymptotic behavior is $\mathcal{O}(n^{1/2} \ln n)$ compared to $\mathcal{O}(\ln n)$.

4. Numerical Results

In this section, some results are given from numerical computation of the improvement factor $M^*/\hat{M}$ defined in Theorem 1. Three different kinds of the weightfunction $\rho(x)$ have been considered.

In Table II, $\rho(x) = (1-x)^\alpha(1+x)^\beta$ and $x \in [-1, 1]$. We choose the net $\{x_i\}_{i=1}^n$ to be zeros of $P_n^{(2\alpha-\frac{1}{2},2\beta-\frac{1}{2})}(x)$, where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial with weightfunction $(1-x)^\alpha(1+x)^\beta$. As we can see, these estimates do not vary much with α and β. We also notice that the improvement factor does not yet agree with the asymptotic expression $1 + (2/\pi) \ln n$, $n \leqslant 90$. For larger n we cannot compute any reliable value. Furthermore, we have not found any improvement factor greater than 5; $\alpha = 0(1)10$, $\beta = O(1)10$.

TABLE II

	Improvement factor			
n	$\alpha = 0, \beta = 0$	$\alpha = 6, \beta = 3$	$\alpha = 12, \beta = 0$	$1 + (2/\pi) \ln n$
10	3.4	3.2	3.5	2.5
20	3.9	3.6	3.9	2.9
50	4.5	4.1	4.2	3.5
70	4.7	4.3	4.4	3.7
90	4.8	4.4	4.5	3.9

In Table III, $\rho(x) = e^{-x^2}$ and $x \in [-\infty, \infty]$. We choose the net $\{x_i\}_{i=1}^n$ to be the zeros of the orthogonal polynomial associated with weightfunction e^{-2x^2} on the interval $[-\infty, \infty]$. We see that the interpolation polynomials are close to the best; the improvement factor is less than 4 if $n \leqslant 20$.

TABLE III

N	Improvement factor
2	2.3
4	2.7
6	3.0
8	3.2
10	3.3
12	3.4
14	3.5
16	3.6
18	3.7
20	3.7

In Table IV, $\rho(x) = e^{-x}$ and $x \in [0, \infty]$. We choose the net $\{x_i\}_{i=1}^n$ to be the zeros of the orthogonal polynomial associated with weightfunction e^{-2x} on the interval $[0, \infty]$. From the table we see that the improvement factor is of modest size.

TABLE IV

N	Improvement factor
2	3.8
4	5.4
6	6.7
8	7.8
10	8.8

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